A topological mechanism of discretization for the electric charge

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Abstract

We present a topological mechanism of discretization, which gives for the fundamental electric charge the value $q_0 = \sqrt{\hbar c} = 5.29 \times 10^{-19}$ C, about 3.3 times the electron charge. Its basis is the following recently proved property of the standard linear classical Maxwell equations: they can be obtained by change of variables from an underlying topological theory, using two complex scalar fields ϕ and θ the level curves of which coincide with the magnetic and the electric lines, respectively.

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The discretization of the electric charge is one of the most important and intriguing laws of physics. However, the value of the fundamental charge is obtained through experiments, all the efforts to predict it — or the fine structure constant α — within a theoretical scheme having failed so far.

A model of electromagnetism proposed a few years ago [1, 2, 3] (to be called "the topological model" from now on) uses the idea of force line as the basic dynamic element. It defines the magnetic and electric lines as the level curves of two complex scalar fields ϕ and θ so that each one is labelled by a complex number. The topology of the force lines plays a very important role and has several curious consequences; this paper shows that one of them is that the model contains only electromagnetic fields which are coupled to electric charges equal to nq_0, n being an integer and $q_0 = \sqrt{\hbar c} = 5.29 \times 10^{-19}$ C, about 3.3 times the electron charge, in such a way that, among the electric lines converging to or diverging from the charge, there are precisely $|n|$ whose label is any prescribed complex number. The model is locally equivalent to Maxwell standard theory, in the sense that any standard field is locally equal to one of the topological fields (so called electromagnetic knots), which shows that Maxwell equations are compatible with the existence of topological charges. However there is a difference of global character referring to the way the fields surround the point at infinity (a natural way in the topological model, allowing only electromagnetic fields of finite energy in empty space, for instance.) All this shows the existence of a beautiful mathematical structure, unknown thus far and which is characteristic of Maxwell standard theory. In this paper, we consider this new idea.

This important law is usually stated by saying that the electric charge of any particle is an integer multiple of a fundamental value e, the electron charge, whose value in the International System of Units is $e = 1.6 \times 10^{-19}$ C. The Gauss theorem allows a different, although fully equivalent, statement of this property: the electric flux across any closed surface Σ which does not intersect any charge is always an integer multiple of e (we will use here the rationalized MKS system). This can be written as

$$\int_\Sigma \omega = ne, \tag{1}$$

where ω is the 2-form $\mathbf{E} \cdot \mathbf{n}\, dS$, $\mathbf{n}$ being a unit vector orthogonal to the surface, $\mathbf{E}$ the electric field and dS the surface element. We could as well write (1) as

$$\int_\Sigma *\mathcal{F} = ne, \tag{2}$$

$*\mathcal{F}$ being the dual to the Faraday 2-form $\mathcal{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. Stating in this way the discretization of the charge is interesting because it shows a close similarity with the expression of the topological degree of a map. Assume that we have a regular map θ of Σ on a 2-sphere S^2 and let σ be the normalized area 2-form in S^2. It then happens that

$$\int_\Sigma \theta^* \sigma = n, \tag{3}$$

$\theta^*\sigma$ being the pull-back of σ and n an integer called the degree of the map, which gives the number of times that S^2 is covered when one runs once through Σ (equal to the number of points in Σ in which θ takes any prescribed value). Note that θ^* in (3) indicates pull-back by the map θ and must not be mistaken for the complex conjugate of θ, which will be written $\bar{\theta}$.

The comparison of (2) and (3) shows that there is a close formal similarity between the dual to the Faraday 2-form and the pull-back of the area 2-form of a sphere S^2. It can be expressed in this way. Let an electromagnetic field be given, such that its form $*\mathcal{F}$ is regular except at the positions of some point charges. Let a map $\theta : R^3 \mapsto S^2$ be also given, which is regular except at some point singularities where its level curves converge or diverge. It happens then that equations (2) and (3) are simultaneously satisfied for all the closed surfaces Σ which do not intersect any charge or singularity.

This means that the electric charge will be automatically and topologically discretized in a model in which these two forms — $*\mathcal{F}$ and $\theta^*\sigma$ — are proportional, the fundamental charge being equal to the proportionality coefficient and the number of fundamental charges in a volume having then the meaning of a topological index.

A way to achieve such a model is the following. Let a complex scalar field $\theta(\mathbf{r}, t)$ be given. Via stereographic projection, the complete complex plane can be identified with the sphere S^2, so that θ can be interpreted as giving a map $\theta : R^3 \mapsto S^2$ at any time. Suppose now that there is an electromagnetic field such that the dual to its Faraday form $*\mathcal{F}$ and the pull-back $\theta^*\sigma$ are proportional, $*\mathcal{F} = \lambda\theta^*\sigma$. As the dimensions of λ are square root of action times velocity, this can be written as

$$*\mathcal{F} = \sqrt{a}\, \theta^* \sigma, \tag{4}$$

where a is a normalizing constant with dimensions of action times velocity. It turns out then that

$$*\mathcal{F} = \sqrt{a}\,\theta^*\sigma = \frac{\sqrt{a}}{2\pi i}\frac{d\theta\wedge d\bar{\theta}}{(1+\bar{\theta}\theta)^2}, \tag{5}$$

the dual to the Faraday tensor being then

$$*F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i}\frac{\partial_\mu\theta\partial_\nu\bar{\theta} - \partial_\nu\theta\partial_\mu\bar{\theta}}{(1+\bar{\theta}\theta)^2}. \tag{6}$$

Note that the electric field is $\mathbf{E} = \sqrt{a}\,(2\pi i)^{-1}(1+\bar{\theta}\theta)^{-2}\nabla\bar{\theta}\times\nabla\theta$ (because of (6)), so that the electric lines are the level curves of θ. The degree of the map $\Sigma \mapsto S^2$ induced by θ is given by (3); it turns out therefore that

$$\int_\Sigma *\mathcal{F} = n\sqrt{a}. \tag{7}$$

As this is equal to the charge Q inside Σ, it does happen that $Q = n\sqrt{a}$, what implies that there is then a fundamental charge $q_0 = \sqrt{a}$, the degree n being the number of fundamental charges inside Σ. This gives a topological interpretation of n.

It is easy to understand that $n = 0$ if θ is regular in the interior of Σ. This is because each level curve of θ (*i. e.* each electric line) is labelled by its value along it — a complex number — and, in the regular case, any one of these lines enters into this interior as many times as it goes out of it. But assume that θ has a singularity at point P, from which the electric lines diverge or to which they converge. If Σ is a sphere around P, we can identify R^3 except P with $\Sigma \times R$, so that the induced map $\theta : \Sigma \mapsto S^2$ is regular. In this case, n need not vanish and is equal to the number of times that θ takes any prescribed complex value in Σ, with due account to the orientation. Otherwise stated, among the electric lines diverging from or converging to P, there are $|n|$ whose label is equal to any prescribed complex number.

This shows a recipe for constructing a model with topological discretization of the charge: just guarantee that (4) (or equivalently (6)) will be satisfied. Opportunely enough, it turns out that the topological model above mentioned is of this kind (see [1, 2, 3, 4] and references therein; the same ideas have inspired a model of ball lightning [5]), which is based on the dynamics of the force lines and makes use of two scalar fields ϕ and θ, interpreted as maps $S^3 \mapsto S^2$ to define the electromagnetic radiation fields through (4)-(6) and the analogous relations

$$\mathcal{F} = -\sqrt{a}\,\phi^*\sigma, \tag{8}$$

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i}\,\frac{\partial_\mu\bar{\phi}\partial_\nu\phi - \partial_\nu\bar{\phi}\partial_\mu\phi}{(1+\bar{\phi}\phi)^2}, \tag{9}$$

As the magnetic field is $\mathbf{B} = -\sqrt{a}\,(2\pi i)^{-1}(1+\bar{\phi}\phi)^{-2}\nabla\bar{\phi}\times\nabla\phi$, the magnetic and electric lines are the level curves of ϕ and θ, respectively.

To understand better this mechanism of discretization, let us take the case of a Coulomb potential [4, 3], $\mathbf{E} = Q\mathbf{r}/(4\pi r^3)$, $\mathbf{B} = 0$. The corresponding scalar is then

$$\theta = \tan\left(\frac{\vartheta}{2}\right)\,\exp\,\left(i\frac{Q}{\sqrt{a}}\varphi\right), \tag{10}$$

where φ and ϑ are the azimuth and the polar angle. The scalar (10) is well defined only if $Q = n\sqrt{a}$, n being an integer. The lines diverging from the charge are labelled by the corresponding value of θ, so that there are $|n|$ lines going in or out of the singularity and having any prescribed complex number as their label. If $n = 1$, it turns out that $\theta = (x+iy)/(z+r)$.

This mechanism has a very curious aspect: it does not apply to the source but to the electromagnetic field itself. This is surprising; one would expect that the topology should operate restricting the fields of the charged particles. However, in this model, it is the field who mediates the force the one which is submitted to a topological condition.

These two scalars are assumed to represent maps $S^3 \mapsto S^2$, which are regular except for singularities at the position of point charges. Consider first the case of empty space, without charges. Four properties of the topological model must be emphasized.

(i) The maps ϕ and θ are dual the one to the other in the sense that (4)-(8) are verified simultaneously (in a more formal notation, this is written $*(\phi^*\sigma) = -\theta^*\sigma$, $*$ being the Hodge or duality operator). Surprisingly, their mere existence implies that the electromagnetic tensor and its dual obey necessarily the equations (with $F_{\mu\nu}$ and $*F_{\mu\nu}$ given by (6) and (9))

$$\partial^\mu F_{\mu\nu}(\bar{\phi},\phi) = 0, \quad \partial^\mu{*F_{\mu\nu}}(\bar{\theta},\theta) = 0, \tag{11}$$

which can be obtained also by an action principle with the usual Lagrangian density $-F_{\mu\nu}F^{\mu\nu}/4$, but taking the two scalars as fundamental fields [2, 3].

This can be stated also as follows. Let us define the product map $\chi = \phi \times \theta : S^3 \mapsto S^2 \times S^2$, so that $\mathcal{V} = \chi^*(\sigma \wedge \sigma) = \phi^*\sigma \wedge \theta^*\sigma$ is the pull-back to S^3 of the volume form in $S^2 \times S^2$. It happens then that $a\,\mathcal{V}/2 = -\mathcal{F}(\phi) \wedge *\mathcal{F}(\theta)/2 = -F_{\mu\nu}F^{\mu\nu}d^4x/4$, if the scalars satisfy the above duality condition. Surprisingly enough, it turns out that the action

$$\mathcal{S} = \frac{a}{2c}\int \mathcal{V}, \tag{12}$$

takes a stationary value for any pair of maps ϕ, θ.

Note that (11) are highly nonlinear in the scalars but become exactly the linear Maxwell equations in the fields $F_{\mu\nu}$ and $*F_{\mu\nu}$. In this sense, the Maxwell equations are the exact linearization (by change of variables, not by truncation!) of a nonlinear theory with topological properties, in which the force lines coincide with the level curves of two scalar fields. The model gives thus a line-dynamics.

(ii) To require that the scalars give regular maps $S^3 \mapsto S^2$ is equivalent to impose on them a condition of compactification: that they have only one value at infinity. This is what allows compactifying the space R^3 to S^3 and has an important consequence: the corresponding Hopf indices are two topological constants of the motion, equal to the linking numbers of any pair of electric or magnetic lines. It happens that these two numbers are equal (say to the integer m) and equal also, up to the factor a, to the common value of the magnetic and electric helicities

$$h_m = \int \mathbf{A}\cdot\mathbf{B}\,d^3r = ma, \quad h_e = \int \mathbf{C}\cdot\mathbf{E}\,d^3r = ma, \tag{13}$$

where $\mathbf{A}$ and $\mathbf{C}$ are vector potentials for $\mathbf{B}$ and $\mathbf{E}$ and the integrals extend to all R^3 (see [2, 6, 7, 8] for discussions on the idea of helicity). The corresponding waves have been called *electromagnetic knots* because any pair of magnetic lines and any pair of electric lines is a link with the same linking number n. In references [9, 10, 3] the explicit expressions for families of such knots are given.

(iii) The electromagnetic knots have the following nice property: any standard radiation electromagnetic field defined in a bound domain of spacetime is locally equal to an electromagnetic knot, except on a zero measure set (for the proof, see section 4 of [2] and section 2 of [3]). In this precise sense, it can be said that the sets of the electromagnetic knots and of the standard radiation fields coincide. As any standard field is locally equal to the sum of two radiation fields, the topological model is locally equivalent to Maxwell standard theory, although they are different globally. This means that one can not distinguish the one from the other by looking only locally as it is done in most experiments. The only difference is of global character and refers to the way in which the fields surround the infinity. (Note incidentally that the Bohm-Aharonov effect which has nonlocal character requires topological considerations.)

(iv) It turns out that the electromagnetic knots verify

$$ma = \hbar c(N_R - N_L), \tag{14}$$

where m is the linking number of any pair of magnetic or electric lines, and N_R, N_L are the classical expressions for the numbers of right and left-handed photons (*i. e.* $N_R = \int \bar{a}_R a_R d^3k$, $N_L = \int \bar{a}_L a_L d^3k$, the functions $\bar{a}_{R,L}, a_{R,L}$ being here Fourier transforms of the classical vector potential $\mathbf{A}$, the c-number fields which are interpreted in QED as creation and annihilation operators for right and left polarization photons. See [3] and references therein for the proof). It must be stressed that the right hand side of (14) is fully meaningful and is well defined as a classical quantity.

Equation (14) relates in a simple an elegant way the two meanings of the word helicity, referring to the wave and particle aspects of the field. At left, the wave helicity ma characterizes the way in which the force lines curl themselves the ones around the others; at right, $\hbar c$ times the particle helicity $N_R - N_L$, to which right- and left-handed photons contribute with $+1$ and -1, respectively. They are equal. Suprisingly this relation arises in a classical context because of topological reasons. The consequence is that $N_R - N_L$ is topologically discretized in the topological model, even in the fields are c-numbers, suggesting that it could give a classical limit with the right normalization. This suggests taking $a = \hbar c$, so that $N_R - N_L = m$. This is surprising: $N_R - N_L$ is then equal to the linking number of the force lines! (and as such it is always an integer number and can take any integer value).

These properties may seem strange in a theory which uses the linear Maxwell equations. However, there is no contradiction, since the electromagnetic fields verifying (6) and (9) form a nonlinear subset of the vector space of the solutions of Maxwell equations, in spite of its set being locally equivalent to the set of the standard fields. This property has been called hidden nonlinearity [2, 10].

All the necessary details about this topological model are explained in the above quoted references [1, 9, 4, 2, 10, 3] (as a sign of the current interest in linked and knotted configurations of classical fields, see reference [11]).

The previous arguments, four properties, and equations (11), (13) and (14) refer to the case of empty space, where the maps ϕ, θ are everywhere regular. To include point charges, the scalar θ must have singularities where its level curves (*i. e.* the electric lines) converge or diverge. As explained above the value of the charge must be then topologically discretized. With $a = \hbar c$, the fundamental charge is

$$q_0 = \sqrt{\hbar c}, \tag{15}$$

(in the MKS system) which is about 3.3 times the electron charge. In the ISU, this is $q_0 = \sqrt{\hbar c \epsilon_0} = 5.29 \times 10^{-19}$ C, and in natural units $q_0 = 1$.

To summarize, in the topological model explained in [1, 9, 4, 2, 10, 3], the magnetic and electric lines are described as the level curves of two complex scalar fields ϕ and θ. As a consequence of the topology of these lines, some integer numbers characterize the electromagnetic fields. In empty space, the fields are called electromagnetic knots, since any pair of magnetic or electric lines are linked with the same linking number m, the magnetic and electric helicities being both $m\hbar c$. It turns out that $m = N_R - N_L$, N_R, N_L being the classical expressions which are interpreted in QED as numbers of right- and left-handed photons. Suprisingly for a topological model, it is locally equivalent to Maxwell theory (because any standard field is locally equal to a field of the model, generated by two scalars), the difference being of global character and referring to the way the fields behave around the point at infinity. Furthermore, the electromagnetic fields can only be coupled to point charges which are integer multiple of the fundamental charge $q_0 = \sqrt{\hbar c}$. Note that the same discretization mechanism would apply to the hypothetical magnetic charges (located at singularities of ϕ), their fundamental value being also $q_0 = \sqrt{\hbar c}$.

Monopoles can be included, therefore, in this model in a natural and simple way, but with a magnetic charge different from the well known Dirac value $g = 2\pi/e$. To be specific, one has in natural units $e = 0.3028$ and $g = 20.75$, so that $e < q_0 < g$. In other words, the fundamental charge q_0 of this model is bigger than the electron charge e, but smaller than the Dirac monopole g. Maybe it should be mentioned that, as the vacuum is dielectric and paramagnetic, the quantum corrections due to the sea of virtual pairs should decrease the electric charge but increase the magnetic one (in other words, the observed electric charge must be smaller than the bare one, the opposite being true for the magnetic charge)

This topological model shows thus a striking electromagnetic duality, since the electric and the magnetic charge have the same value $\sqrt{\hbar c}$ at the classical level (at which this model applies in its present form and where there are no vacuum polarization effects) and, moreover, they are inverse to one another, in natural units. Consequently, if the sea of virtual pairs is included, the observed electric and magnetic charges must be smaller and bigger than $\sqrt{\hbar c}$, respectively, in agreement with current knowledge, since the Dirac monopole must be understood as a dressed magnetic charge.

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